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Corrections to the wave function and the hyperfine structure in exotic atoms

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Abstract. We present an analytical evaluation of radiative corrections in exotic atoms induced by the one–loop electronic vacuum polarization. We evaluate corrections to the energy levels, to the wave function (at the origin) and to the hyperfine structure. We treat all corrections analytically within a non-relativistic approximation. Agreement is found with a few available numerical results. The analytical treatment allows to determine the asymptotic forms of the corrections in the limit of a small atomic radius, which for the atomic systems considered corresponds to a large mass of the constituent particle as compared to the electron mass. The asymptotics can be verified using the effective charge approach.

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1 Introduction

Exotic atoms, whose constituent particles are heavier than the electron, share characteristics with both weakly bound hydrogen–like atoms and high–Z electronic systems. For low charge the exotic bound systems are non-relativistic (like weakly bound systems), because the characteristic atomic momentum is of the order of $\alpha\, m_R$ (where m_R is the reduced mass of the system). By contrast, the atomic radius of exotic atoms is small and comparable to that of strongly bound electronic systems. In muonic hydrogen, *e.g.*, the Bohr atomic radius is equal to $1/(\alpha\, m_\mu)$ (ignoring reduced mass corrections). This radius is roughly 207 times smaller than the atomic radius of hydrogen and smaller than the radius of typical strongly bound electronic systems (*e.g.* hydrogen–like uranium). Because of this small length scale, the correction to almost all atomic quantities of interest in exotic systems caused by the free electronic vacuum polarization is significant in the muonic atoms and exotic system, but it can be neglected in "usual" electronic atoms at low Z in the next-to-leading order. In this paper we derive analytical expressions for three corrections,

- as a starting point the energy shift ΔE of low-lying atomic levels induced by the Uehling potential (the dominant radiative *correction to the energy*),
- the correction ΔE_F to the hyperfine structure by a vacuum polarization insertion in the transverse photon line (*transverse hyperfine structure correction*),
- the correction to the value of the wave function at the origin $\Delta\psi(0)$ (*wave function correction*).

The analytical expressions are evaluated for pionium ($\pi^+\pi^-$-atom), dimuonium (bound $\mu^+\mu^-$–system) and for the hypothetical tauonium particle ($\tau^+\tau^-$–atom). We work in this article within a non relativistic approximation.

Of particular importance for the physical properties of exotic atoms is the correction to the wave function at the origin $\Delta\psi(0)$. It should be noted that the wave function correction contributes, like the transverse correction and other contributions, to the hyperfine structure. However, the meaning of the wave function correction is rather universal. The Fermi energy, the finite nuclear size correction to the energy levels, the decay rate of pionium to neutral pions and the annihilation decay rate of dimuonium are all proportional to the square of the wave function at the origin, $|\psi(0)|^2$. In first approximation, the value of the wave function at the origin is $|\psi(0)|^2 = (Z\,\alpha\, m_R)^3/(\pi\, n^3)\,\delta_{l,0}$ and vanishes for P states and for states of higher angular momenta. A radiative correction to the value of $\psi(0)$, which will be considered only for S states in this paper for obvious reasons, causes a modification of all

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quantities which are proportional to it. *E.g.*, the decay rate is modified according to

$$\left[\frac{\Delta\Gamma}{\Gamma}\right]_{\psi} = 2\,\frac{\Delta\psi_{\rm nS}(0)}{\psi_{\rm nS}(0)}\ . \tag{1}$$

The wave function correction due to the electronic vacuum polarization in exotic atoms is of relative order α/π and therefore larger than any relativistic corrections. The correction is in first approximation spin–independent and universal to all heavy exotic atoms consisting of bosons and/or fermions. In reference [4] we evaluate the wave function correction numerically, but we restrict the discussion to dimuonium and we do not discuss the asymptotical behaviour. In this paper, we report results for pionium and tauonium and present a detailed discussion of the asymptotical behaviour of the wave function correction, which illustrates its physical origin.

The relative order–of–magnitude of the wave function correction for exotic atoms (α/π) can be derived as follows. For light electronic systems, the dominant correction to the wave function is of relative order $(\alpha/\pi)\,(Z\,\alpha)^2$ at distances of the order of $1/(Z\alpha m_R)$ and of relative order $(\alpha/\pi)(Z\alpha)$ at smaller distances of the order $1/m_R$. By contrast, for heavier systems, the two factors of $(Z\,\alpha)$ are compensated by the smaller length scale of the heavy atom as compared to the Compton wavelength of the electron, and the correction enters at the level of α/π. It turns out that virtual free intermediate states of momentum of the order of the electronic mass yield the most significant contribution to the wave function correction.

We would like to conclude this Introduction with a brief discussion of the possible experimental realizations of pionium and dimuonium. The dimuonic system could be produced in heavy–ion collisions and in electron–positron colliders. The possibility of production of this particle in heavy–ion collisions at RHIC and LHC energies and luminosities has been estimated with promising results [1]. Pionium has been observed recently in a remarkable experiment [2]. Radiative corrections to its decay are studied in reference [3].

2 Energy correction

The one–loop vacuum polarization correction to the photon propagator can be represented as a replacement of the form

$$\frac{1}{q^2+i\,\epsilon} \to \frac{\alpha}{\pi}\int_{s_0}^{\infty} ds\ \rho(s)\,\frac{1}{q^2-s+i\,\epsilon}\,, \tag{2}$$

where s_0 is the threshold of pair production of the particle in the vacuum polarization loop. For electronic vacuum polarization, $s_0 = 4\,m_e^2$, where m_e is the mass of the electron, and the spectral function reads (see *e.g.* Ref. [5])

$$\rho(s) = \frac{1}{3\,s}\,\sqrt{1-\frac{4\,m_e^2}{s}}\,\left(1+\frac{2\,m_e^2}{s}\right)\,.$$

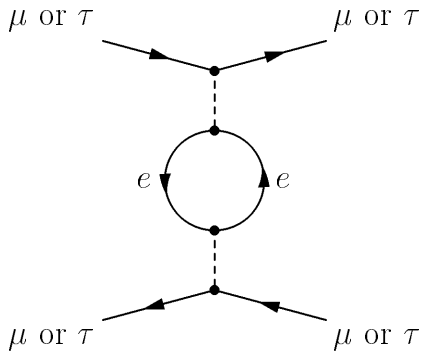

Fig. 1. Feynman diagram for the energy correction due to electronic vacuum polarization, which has a big effect in heavy particle–antiparticle systems (the diagram is for heavy fermionium).

With the substitution $v^2 = 1 - 4\,m_e^2/s$ the one-loop vacuum polarization integral can be cast into the form [6]

$$\frac{1}{q^2+i\,\epsilon} \to \frac{\alpha}{\pi}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{1-v^2}\,\frac{1}{q^2-\lambda^2+i\,\epsilon}\,,$$

where

$$\lambda \equiv \lambda(v) = \frac{2\,m_e}{\sqrt{1-v^2}}\,.$$

In non-relativistic approximation, we can restrict ourselves to purely space-like momentum transfer. The resulting correction to the Coulomb potential, which is caused by a momentum transfer with $q^0 = 0$, (we have $V(\mathbf{k}) = -(4\,\pi\,Z\,\alpha)/\mathbf{k}^2$), can be Fourier-transformed into coordinate space and results in the Uehling potential,

$$V_U(\mathbf{r}) = \frac{\alpha}{\pi}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{1-v^2}\,\left[\frac{-Z\,\alpha}{r}\right]\,\exp\left(-\lambda\,r\right)\,. \tag{3}$$

In heavy fermionium and pionium, the charge Z of the particles is equal to one, and we will assume that Z is equal to one or small (less than 20) in the sequel. From equation (3) it follows that in leading approximation the correction to the energy is spin–independent. It is represented diagrammatically in Figure 1. We evaluate the energy shift given by the matrix element

$$\Delta E = \langle\psi|V_U|\psi\rangle = \frac{\alpha}{\pi}\,C_E\,E_\psi\,, \tag{4}$$

where C_E is a dimensionless coefficient and

$$E_\psi = -\frac{(Z\,\alpha)^2\,m_R}{2\,n^2} \tag{5}$$

is the Schrödinger binding energy. The coefficients C_E are all positive, which is consistent with the attractive nature of the vacuum polarization correction to the Coulomb potential. It is interesting to note that according to equation (4), the energy shift due to vacuum polarization, which enters at relative order α/π, has a greater effect than for

example, any relativistic correction of order $(Z\alpha)^2$ and higher. The evaluation of the matrix element in equation (4) proceeds as follows. We perform all calculations in coordinate space. Angular integration is trivial, and the radial integration can be performed using standard integrals. For the remaining integration over the spectral function of vacuum polarization, the substitution $v \to \sin\varphi$ proves to be useful. The remaining integrals can be found in reference [7]. Identification of the (state–dependent) parameter

$$\kappa_n = \frac{\kappa}{n} = \frac{Z\,\alpha\,m_R}{m_e\,n} \tag{6}$$

allows to rewrite the result in compact form. It is of interest to evaluate κ_n for the systems considered in this paper. We obtain,

$$\kappa(\mu^+\mu^-) = 0.754, \quad \kappa(\pi^+\pi^-) = 0.997$$
$$\text{and} \quad \kappa(\tau^+\tau^-) = 12.7\,. \tag{7}$$

In this article, we will defer the discussion of excited states to the Appendices. Here, we will implicitly assume that results are conferred for the $1S$ ground state and use the fact that $\kappa = \kappa_1$. We obtain the analytical expression,

$$C_E(1S) = \pi\,\left(\frac{4}{3\,\kappa^3} + \frac{1}{\kappa}\right) - \left(\frac{8}{3\,\kappa^2} + \frac{22}{9}\right) + \frac{2\,\left(2\,\kappa^4 - \kappa^2 - 4\right)}{3\,\kappa^3}\,\frac{\arccos\kappa}{\sqrt{1-\kappa^2}}\,, \tag{8}$$

where C_E is implicitly defined in equation (4). The analytical continuation of the expression in equation (8) is given by

$$\frac{\arccos\kappa}{\sqrt{1-\kappa^2}} = \frac{\ln\left(\kappa + \sqrt{\kappa^2-1}\right)}{\sqrt{\kappa^2-1}} \quad \text{for} \quad \kappa > 1\,.$$

The result in equation (8) has been obtained earlier by Pustovalov [8] in closed form. We confirm the result in reference [8] and additionally analyze the asymptotical form of the energy correction here. It should be pointed out that the analysis of the energy correction serves as a preliminary step towards the treatment of the hyperfine structure correction and the wave function correction. From equation (8), we obtain the following asymptotics in the limit of a small atomic radius,

$$C_E(1\mathrm{S}) \sim \frac{4}{3}\,\ln(2\,\kappa) - \frac{22}{9} \quad \text{for} \quad \kappa \to \infty\,. \tag{9}$$

The limit of small atomic radius corresponds to a large mass of the constituent particle in the particle–antiparticle systems as compared to the electron mass, not the relativistic limit of high Z (*cf.* Eq. (6)). Z must be kept small (less than 20) to insure the non–relativistic nature of the system. This limit of small atomic radius (large mass ratio) is realized to a certain extent in the hypothetical tauonium system (*cf.* Eq. (7)) and in muonic atoms with moderate nuclear charge number Z. *E.g.*, we obtain for the $1S$ state in tauonium,

$$C_E(1S, \tau^+\tau^-) \simeq 1.87$$
(from the asymptotic form, Eq. (9))

Table 1. C_E coefficients for the energy shifts in exotic atoms are tabulated for dimuonium, tauonium and pionium.

C_E	dimuonium	pionium	tauonium
$1S$	0.15	0.22	2.10
$2S$	0.071	0.10	1.05
$2P$	0.0017	0.0041	0.63

and

$$C_E(1S, \tau^+\tau^-) = 2.10$$
(from the exact expression, Eq. (8))

within roughly 10% agreement. For the limit $\kappa \to 0$ we obtain

$$C_E(1S) \sim \frac{8\,\kappa^2}{15} \quad \text{for} \quad \kappa \to 0\,. \tag{10}$$

Equation (10) confirms the contribution of order $(\alpha/\pi)\,(Z\,\alpha)^4$ to the Lamb shift in electronic systems induced by vacuum polarization. Results for the energy correction for $1S$, $2S$ and $2P$ states are given in Appendix A and summarized in Table 1.

The leading logarithmic coefficient in equation (9) can be obtained with the effective charge approach, which is discussed in more detail in [9]. In leading approximation, one may replace the electromagnetic coupling upon insertion of one electronic vacuum polarization loop by

$$\text{coupling} \to \text{coupling} \times \left[1 + \frac{\alpha}{3\,\pi}\,\ln\left(\frac{q^2}{m_e^2}\right)\right]$$
$$\text{for} \quad q^2 \to \infty \quad \text{(in logarithmic approximation)} \tag{11}$$

where q^2 is a typical photon four–momentum for the correction under consideration. The atomic momentum $q_{atom} = Z\,\alpha\,m_R$ is characteristic for a particle bound in a Coulomb potential. Because the Schrödinger energy is proportional to $(Z\,\alpha)^2$ (see Eq. (5)), the effective charge approach according to equation (11) calls for a replacement

$$(Z\,\alpha)^2 \to (Z\,\alpha)^2 \times \left[1 + 2\,\frac{\alpha}{3\,\pi}\,\ln\left\{\left(\frac{Z\,\alpha\,m_R}{m_e}\right)^2\right\}\right]$$
$$\simeq (Z\,\alpha)^2 \times \left[1 + \frac{4\,\alpha}{3\,\pi}\,\ln\kappa\right] \quad \text{for} \quad \kappa \to \infty \tag{12}$$

in agreement with equations (9, 28, 31).

3 Transverse hyperfine structure correction

One of the terms in the Breit Hamiltonian which generate the hyperfine structure originates from the exchange of a transverse photon. This term for S states is proportional to $(\mathbf{S}_1 \cdot \mathbf{S}_2)\,\nabla^2 V(r)$. $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin operators of

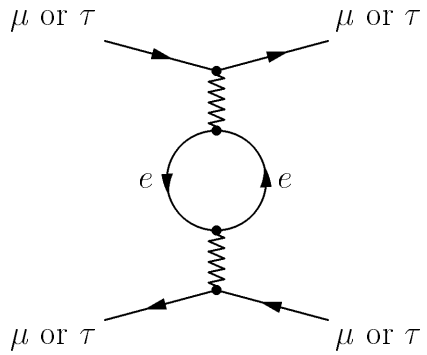

Fig. 2. Feynman diagram for the vacuum polarization correction to the transverse photon in heavy fermionium. This process contributes to the hyperfine structure. The correction is referred to as the *transverse hyperfine structure correction* in the text.

Table 2. C_T coefficients for the transverse photon–vacuum polarization correction to the hyperfine splitting in heavy fermionium.

C_T	dimuonium	tauonium
$1S$	0.60	2.27
$2S$	0.62	2.31

the two particles, and $V(r) = -Z\,\alpha/r$ is the binding potential. In the absence of virtual annihilation processes, this term generates the hyperfine splitting in lowest order (given by the Fermi energy). The correction induced by a vacuum polarization insertion in the transverse photon is proportional to $(\mathbf{S}_1 \cdot \mathbf{S}_2)\,\nabla^2 V_U(r)$ where V_U is the Uehling potential (Eq. (3)). For a compact presentation of the result, it customary to define

$$\Delta E_F = \frac{\alpha}{\pi}\, C_T\, E_F^{(T)}\,, \tag{13}$$

where $E_F^{(T)}$ is the portion of the Fermi energy originally caused by the transverse photon exchange diagram (see Fig. 2). For dimuonium and positronium, virtual annihilation also contributes to E_F. The portion of E_F caused by transverse photon exchange is $E_F^{(T)} = (4/7)\,E_F$. By contrast, for pionium, there is no hyperfine splitting due to the spin–0 character. The evaluation of the correction in equation (13) can be done analytically. It is carried out in momentum space, where the Laplacian operator assumes a simple form. Angular integration is trivial, and the momentum space integral can be evaluated by joining denominators with suitable (Feynman) parameters. For the final integration over the spectral function of vacuum polarization, the same substitution as for the energy correction proves to be useful. The result is given here in terms of the dimensionless constant C_T,

$$C_T(1S) = \pi\left(-\frac{1}{3\,\kappa^3}\right) + \frac{6+\kappa^2}{9\,\kappa^2} + \frac{2-\kappa^2+2\,\kappa^4}{3\,\kappa^3}\,\frac{\arccos\kappa}{\sqrt{1-\kappa^2}}\,. \tag{14}$$

The result for the hyperfine structure correction for the $2S$ state is presented in Appendix B and summarized for the atomic systems of interest in Table 2. The asymptotic forms of the correction are

$$C_T(1S) \sim \frac{2}{3}\,\ln(2\,\kappa) + \frac{1}{9} \quad \text{for} \quad \kappa \to \infty \tag{15}$$

and

$$C_T(1S) \sim \frac{3\,\pi\,\kappa}{8} \quad \text{for} \quad \kappa \to 0\,. \tag{16}$$

The leading logarithmic coefficient in equation (15) can be obtained easily within the effective charge approach by considering the modification of the binding factor $Z\,\alpha$ in the Uehling potential (only one factor of $Z\,\alpha$ contributes in this case). The effective charge approach therefore yields

$$\begin{aligned}(Z\,\alpha) &\to (Z\,\alpha)\times\left[1+\frac{\alpha}{3\,\pi}\,\ln\left\{\left(\frac{Z\,\alpha\,m_R}{m_e}\right)^2\right\}\right]\\ &\simeq (Z\,\alpha)\times\left[1+\frac{2\,\alpha}{3\,\pi}\,\ln\kappa\right] \quad \text{for} \quad \kappa\to\infty\end{aligned} \tag{17}$$

in agreement with equation (15). For the $2S$ state, we consistently find the same logarithmic coefficient as for $1S$ (see Eq. (34)). Results obtained by evaluation of equation (14) are in excellent agreement with a numerical calculation of Sternheim [10] for various muonic atoms with nuclear charges $Z = 1$ ($\kappa \approx 1.4$) to $Z = 9$ ($\kappa \approx 13$). For muonic atoms with Z in the range of $4 \le Z \le 9$, the asymptotic form equation (15) reproduces the exact results obtained from equation (14) to within 1%. The κ–values for these atoms approximately range from 6 to 13.

4 Wave function correction

The diagram which induces the wave function correction is depicted in Figure 3 for heavy fermionium (dimuonium or tauonium). The correction enters at the level of α/π. We can write the wave function correction as

$$\frac{\Delta\psi_{\rm nS}(0)}{\psi_{\rm nS}(0)} = -\frac{\langle\psi_{\rm nS}|\delta(\mathbf{r})\,\overline{G}(E_\psi)\,V_U|\psi_{\rm nS}\rangle}{\langle\psi_{\rm nS}|\delta(\mathbf{r})|\psi_{\rm nS}\rangle}\,, \tag{18}$$

where $\overline{G} = 1/(H-E)'$ denotes the reduced Green's function (the reference state is excluded), and V_U denotes the Uehling potential. Angular integration is trivial, and the radial integration can be performed using the explicit representation of the reduced Schrödinger–Coulomb Green's function $\overline{G}(\mathbf{r},\mathbf{r}'=0;E_{ns})$ given in reference [12,13]. The

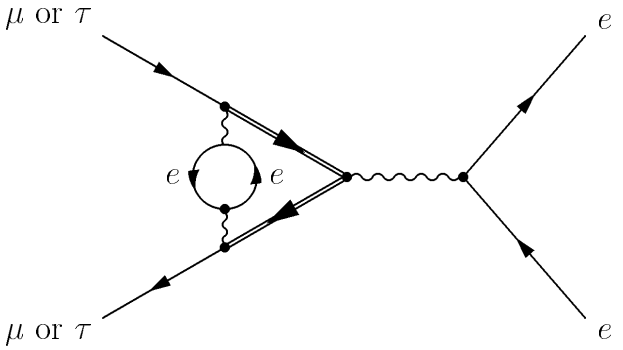

Fig. 3. The wave function correction is represented here by its effect on the decay rate of heavy fermionium into electrons. In this case, the correction is mediated by a vertex correction to the incoming annihilation vertex, with a vacuum polarization insertion in the virtual photon. Double lines denote bound propagators. As it is further explained in the text, the meaning of the wave function correction is rather general.

last step is the integration over the spectral function of vacuum polarization. It is performed with the same substitution $v \to \sin\varphi$ as for the other corrections considered in this paper. For a more compact presentation of the result, it is customary to define

$$\Delta\psi(0) = \frac{\alpha}{\pi}\, C_\psi\, \psi(0)\,, \tag{19}$$

where C_ψ is a dimensionless coefficient. We obtain for the $1S$ state,

$$C_\psi(1S) = \pi\,\frac{\kappa^2-2}{4\kappa^3} + \frac{6-8\kappa^2+5\kappa^4}{6\,\kappa^2\,(1-\kappa^2)} + \frac{2-4\,\kappa^2+3\,\kappa^4-2\,\kappa^6}{2\,\kappa^3\,(1-\kappa^2)}\,\frac{\arccos(\kappa)}{\sqrt{1-\kappa^2}} + J(\kappa)\,. \tag{20}$$

The integral $J(\kappa)$ is given by

$$J(\kappa) = \int_0^1 dy\,\frac{y\,\sqrt{1-y^2}\,(2+y^2)}{3}\,\frac{\kappa^2}{(1+y\,\kappa)^2} \times \ln\left(\frac{1+y\,\kappa}{y\,\kappa}\right)\,. \tag{21}$$

$J(\kappa)$ can be determined analytically, but the result is rather lengthy and contains derivatives of generalized hypergeometric ${}_3F_2$–functions with respect to parameters. We prefer not to indicate the analytical result here. The most compact representation for $J(\kappa)$ is probably given by equation (21). One can nevertheless extract useful information on $J(\kappa)$ from equation (21). For $\kappa \to \infty$, the integrand of $J(\kappa)$ tends to zero, but this convergence is not uniform in the range $[0,1]$. The limit is

$$\lim_{\kappa\to\infty} J(\kappa) = \frac{\pi^2}{9} - \frac{2}{3}\,.$$

We therefore obtain the asymptotic behaviour for $\kappa \to \infty$ of the wave function correction as

$$C_\psi(1S) \sim \ln(2\,\kappa) - \frac{3}{2} + \frac{\pi^2}{9} \quad \text{for} \quad \kappa \to \infty \tag{22}$$

Table 3. C_ψ coefficients for the wave function correction induced by the Uehling potential in a number of exotic atomic systems.

C_ψ	dimuonium	pionium	tauonium
$1S$	0.53	0.68	2.84
$2S$	0.46	0.57	2.15

and

$$C_\psi(1S) \sim \frac{3\,\pi\,\kappa}{16} \quad \text{for} \quad \kappa \to 0\,. \tag{23}$$

The coefficient of the logarithm in equation (22) can be understood from the effective charge approach by considering the unperturbed value of the wave function at the origin, which is proportional to $(Z\,\alpha\,m_R)^{3/2}$. The effective charge prescription equation (11) then yields

$$(Z\,\alpha)^{3/2} \to (Z\,\alpha)^{3/2} \times \left[1 + \frac{3}{2}\,\frac{\alpha}{3\,\pi}\,\ln\left\{\left(\frac{Z\,\alpha\,m_R}{m_e}\right)^2\right\}\right] \simeq (Z\,\alpha) \times \left[1 + \frac{\alpha}{\pi}\,\ln\kappa\right] \quad \text{for} \quad \kappa \to \infty \tag{24}$$

in agreement with equation (22). The result for the $2S$ state is given in Appendix C. For dimuonium, the asymptotic form of the correction (Eqs. (22,37)) cannot be applied, because of the smallness of κ (Eq. (7)). For tauonium, however, the asymptotic form of the correction provides a result correct to the level of 5 per mille. We have from the asymptotic expression

$$C_\psi(1S, \tau^+\tau^-) \simeq 2.83\,\frac{\alpha}{\pi} \quad \text{(from the asymptotic form, Eq. (22))} \tag{25}$$

in good agreement with the full result

$$C_\psi(1S, \tau^+\tau^-) = 2.84\,\frac{\alpha}{\pi} \quad \text{(from the exact expression, Eq. (20))}\,. \tag{26}$$

In addition, the asymptotic given in equation (22) reproduces very well the exact results from equation (20) for muonic atoms with moderate nuclear charge numbers $Z \geq 4$. This Z–range corresponds approximately to $\kappa \geq 6$.

The difference between the asymptotic and the exact result for $Z \geq 4$ in muonic atoms is on the level of 1%. The results for the wave function correction are presented in Table 3. The numerical value for pionium is

$$C_\psi(1S, \pi^+\pi^-) = 0.68\,\frac{\alpha}{\pi} \quad \text{(from Eq. (20))}\,.$$

For pionium, the wave function correction to the best of our knowledge has not been indicated in the literature.

Our results in equations (20, 36) are in excellent agreement with the numerical treatment of the problem in reference [10]. They are, however, in disagreement with reference [11]. In reference [11], the wave function correction

Table 4. Large κ asymptotics for all corrections considered in this paper.

$\kappa \to \infty$	$1S$	$2S$
C_E	$\frac{4}{3}\ln(2\kappa) - \frac{22}{9}$	$\frac{4}{3}\ln(\kappa) - \frac{28}{9}$
C_T	$\frac{2}{3}\ln(2\kappa) + \frac{1}{9}$	$\frac{2}{3}\ln(\kappa) + \frac{11}{18}$
C_ψ	$\ln(2\kappa) - \frac{3}{2} + \frac{\pi^2}{9}$	$\ln(\kappa) - \frac{8}{3} + \frac{2\pi^2}{9}$

is analyzed for dimuonium and tauonium. Results are obtained within some approximations which are valid only in the limit of small κ. A part of these approximations consists in the use of the free Green's function instead of the bound propagator. This is valid in the limit of small κ. Consequently, the value conferred in reference [11] for muonic vacuum polarization in tauonium (small κ parameter) is in agreement with our result. The value of κ in this case is $\alpha\, m_\tau/(2\, m_\mu) = 0.06$. In addition, the asymptotics in equations (23, 38) for $\kappa \to 0$ are in agreement with those found in reference [11] for the limit of small κ. For the *electronic* vacuum polarization in tauonium, however, the κ parameter is large ($\kappa = 12.7$, *cf.* Eq. (7)). Therefore, we obtain a substantial disagreement between our result and the value obtained in reference [11]. In reference [11] a result of $C_\psi(1S, \tau^+\tau^-) = 4.57$ is reported for the $1S$ state in tauonium in contradiction to our result in equation (26).

5 Conclusion

We obtained analytical results for the energy correction, the transverse hyperfine structure correction and the wave function correction in exotic atoms. Results are given in equations (8, 14, 20, 27, 30, 33, 36), and in the Tables 1, 2 and 3. We analyzed the asymptotic behaviour of the corrections and derived leading logarithms in the effective charge approach. The asymptotics in the limit of a small atomic radius (large mass ratio) ($\kappa \to \infty$, *cf.* Eq. (6)) are given in equations (9, 15, 22, 28, 34, 37) and are summarized in Table 4. The state–independent logarithmic coefficients are obtained in equations (12, 17, 24) within the effective charge approach. This derivation illustrates their physical origin (running coupling constant).

We verified numerical results obtained by Sternheim [10] for a number of low-Z muonic atoms and by Pachucki (for muonic hydrogen, [14]) within our analytical treatment. We obtained for the wave function correction to the decay rate of the $\pi^+\pi^-$–atom $\Delta\Gamma(1S)/\Gamma^{(0)}(1S) = 1.36\,(\alpha/\pi)$ and $\Delta\Gamma(2S)/\Gamma^{(0)}(2S) = 1.14\,(\alpha/\pi)$ for pionium (*cf.* Eqs. (1, 20, 36)). Numerical values are also provided for the effect in heavy fermionium (Tab. 3). The asymptotic results derived for the limit of a small atomic radius (large mass ratio) are in fair agreement with a complete evaluation for the heaviest of the systems considered, the hypothetical tauonium particle (*cf.* Eqs. (25) and (26)) and for muonic atoms with moderate nuclear charge numbers Z ($4 \le Z \le 9$), which are considered in the work of Sternheim [10].

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Appendix A: Energy correction for 2S and 2P

In this Appendix we consider the energy shift of the the $2S$ and $2P$ states due to the Uehling potential. We obtain the following results,

$$\begin{aligned} C_E(2S) = \pi\, & \frac{2\,\left(14 + 3\,\kappa_2^2\right)}{3\,\kappa_2^3} \\ & + \frac{-168 + 272\,\kappa_2^2 - 49\,\kappa_2^4 - 28\,\kappa_2^6}{9\,\kappa_2^2\,(\kappa_2^2-1)^2} \\ & + \frac{-56 + 128\,\kappa_2^2 - 75\,\kappa_2^4 - 10\,\kappa_2^6 + 4\,\kappa_2^8}{3\,\kappa_2^3\,(\kappa_2^2-1)^2}\,\frac{\arccos\kappa_2}{\sqrt{1-\kappa_2^2}} \end{aligned} \quad (27)$$

where κ_n is defined in equation (6). We find for the asymptotics

$$C_E(2S) \sim \frac{4}{3}\ln(2\,\kappa_2) - \frac{28}{9} \quad \text{for} \quad \kappa_2 \to \infty \quad (28)$$

and

$$C_E(2S) \sim \frac{\alpha}{\pi}\,\frac{16\,\kappa_2^2}{15} \quad \text{for} \quad \kappa_2 \to 0\,. \quad (29)$$

For the $2P$ state the following results are obtained,

$$\begin{aligned} C_E(2P) = \pi & \frac{2\,\left(10 + 3\,\kappa_2^2\right)}{3\,\kappa_2^3} \\ & + \frac{-120 + 184\,\kappa_2^2 - 23\,\kappa_2^4 - 32\,\kappa_2^6}{9\,\kappa_2^2\,(\kappa_2^2-1)^2} \\ & + \frac{-40 + 88\,\kappa_2^2 - 45\,\kappa_2^4 - 10\,\kappa_2^6 + 4\,\kappa_2^8}{3\,\kappa_2^3\,(\kappa_2^2-1)^2}\,\frac{\arccos\kappa_2}{\sqrt{1-\kappa_2^2}} \end{aligned} \quad (30)$$

with the asymptotics

$$C_E(2P) \sim \frac{4}{3}\ln(2\,\kappa_2) - \frac{32}{9} \quad \text{for} \quad \kappa_2 \to \infty \quad (31)$$

and

$$C_E(2P) \sim \frac{\alpha}{\pi}\,\frac{8\,\kappa_2^4}{35} \quad \text{for} \quad \kappa_2 \to 0\,. \quad (32)$$

From these expressions we deduce the asymptotic form

$$\mathcal{L} = \Delta E(2S - 2P) \sim -\frac{1}{18}\,\frac{\alpha}{\pi}\,(Z\,\alpha)^2\,m_R \quad \text{for} \quad \kappa \to \infty$$

for the Lamb shift $\mathcal{L}$.

Appendix B: Transverse hyperfine structure correction for the 2S state

For the $2S$ state we obtain

$$C_T(2S) = \pi\left(-\frac{1}{3\,\kappa_2^3}\right) + \frac{24-44\,\kappa_2^2-29\,\kappa_2^4+22\,\kappa_2^6}{36\,\kappa_2^2\,(\kappa_2^2-1)^2} + \frac{8-20\,\kappa_2^2+33\,\kappa_2^4-20\,\kappa_2^6+8\,\kappa_2^8}{12\,\kappa_2^3\,(1-\kappa_2^2)^2}\,\frac{\arccos\kappa_2}{\sqrt{1-\kappa_2^2}} \tag{33}$$

as result for the hyperfine structure correction. The asymptotic forms are

$$C_T(2S) \sim \frac{2}{3}\,\ln(2\,\kappa_2) + \frac{11}{18} \quad \text{for} \quad \kappa_2 \to \infty \tag{34}$$

and

$$C_T(2S) \sim \frac{\alpha}{\pi}\,\frac{3\,\pi\,\kappa_2}{4} \quad \text{for} \quad \kappa \to 0\,. \tag{35}$$

Appendix C: Wave function correction for 2S

For the $2S$ state we obtain the following result for the wave function correction,

$$C_\psi(2S) = \pi\,\frac{3\,\kappa_2^2-26}{6\,\kappa_2^3} + \frac{312-920\kappa_2^2+894\kappa_2^4-195\,\kappa_2^6+44\,\kappa_2^8}{36\,\kappa_2^2\,(1-\kappa_2^2)^3} + \frac{104-376\,\kappa_2^2+506\,\kappa_2^4-309\,\kappa_2^6+42\,\kappa_2^8-12\,\kappa_2^{10}}{12\,\kappa_2^3\,(1-\kappa_2^2)^3} \times \frac{\arccos(\kappa_2)}{\sqrt{1-\kappa_2^2}} + K(\kappa_2)\,, \tag{36}$$

where the integral K is given by

$$K(\kappa_2) = \int_0^1 dy\,\frac{2\,y\,\sqrt{1-y^2}\,(2+y^2)}{3}\,\frac{\kappa_2^2\,(2+y^2\,\kappa_2^2)}{(1+y\,\kappa_2)^4} \times \ln\left(\frac{1+y\,\kappa_2}{y\,\kappa_2}\right)\,.$$

K can be obtained by acting of a suitable differential operator on J (Eq. (21)). We easily find the limit

$$\lim_{\kappa\to\infty} K(\kappa_2) = \frac{2\,\pi^2}{9} - \frac{13}{9}\,.$$

The asymptotic behaviour of the correction for the $2S$ state for large κ is

$$C_\psi(2S) \sim \ln(2\,\kappa_2) - \frac{8}{3} + \frac{2\,\pi^2}{9} \quad \text{for} \quad \kappa \to \infty\,, \tag{37}$$

and for small κ it results

$$C_\psi(2S) \sim \frac{3\,\pi\,\kappa_2}{8} = \frac{3\,\pi\,\kappa}{16} \quad \text{for} \quad \kappa \to 0\,. \tag{38}$$

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